NOTES

The Wallet Paradox Revisited

MAUREEN T. CARROLL
University of Scranton
Scranton, PA 18510
carrollm1@uofs.edu

MICHAEL A. JONES
Montclair State University
Upper Montclair, NJ 07043
jonesma@pegasus.montclair.edu

ELYN K. RYKKEN
Muhlenberg College
Allentown, PA 18104
elrykken@muhlenberg.edu

Put your wallet on the table next to mine. The game is this: The person whose wallet has less money wins all the money in the other person's wallet. Do you want to play?

You might think along these lines: "I don't know how much money that other wallet has, and I'm not even sure how much is in mine. If I have more money, then I'll lose it, but if I have less, I'll win the larger amount. I have no idea what the odds are, but since I stand to win more money than I can lose, it seems like a good game." Upon further thought, you realize that both players are probably thinking the same thing! Can both be correct?

How can a game favor both players? It can't! In any two-person, zero-sum game (where one person wins what the other person loses), it is not possible for the game to be advantageous to both players. Believing that the wallet game favors both players is a paradox, one discussed by Martin Gardner [3]. A variation of this game was originally posed by Kraitchik [1] where the person with the greater amount in her wallet gives the difference to the other.

What if you and I decide to play this game day after day? We will need to establish a few more rules, because it would not be a very interesting game if neither of us ever carried any money. Since we do not want to mandate a minimum amount that must be carried, we agree that on average (and in the long run) we will carry the same amount of money. How should you decide how much to carry each day?

Kraitchik shows that if the amount of money each player carries is uniformly (discretely) distributed between 0 and some large x (he uses the total amount of money that has been minted to date), then the game is *fair* (each player's expected payoff is zero). Gardner notes that this does not explain the source of the paradox. Merryfield, Viet, and Watson [2] argue that the source of the apparent paradox is that the players do not take into account the probabilities of winning and losing. They argue that if the amounts of money in the players' wallets are determined by independent, identically distributed random variables, then the game is also fair. Hence, the game is fair when the players are required to use the same distributions.

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It is natural to ask if requiring the players to carry the same amount on average might also ensure a fair game. When "on average" is interpreted as requiring both players' distributions to have the same mean, Merryfield, *et al.* point out that the game may not be fair. In fact, they give an example that shows it is possible to have a smaller mean than one's opponent and still be at a disadvantage (that is, have a negative expected payoff). At the end of their paper, they pose the following question:

> If we suppose that the distributions of players A and B are required to have the same means, is there a strategy that player A could adopt to have a winning edge? In other words, is there a preferred distribution (or a winning strategy)? [2]

The answer to this question depends upon whether knowledge of an opponent's strategy, *not just the mean*, is assumed. In this article, we show that if a player knows her opponent's strategy, then she can construct a winning strategy which has the same positive mean or median as her opponent. This implies that there is no optimal strategy (or Nash equilibrium) when players are restricted to use strategies with the same mean or median. We consider both the discrete and continuous cases. Throughout, we assume that players' distributions are independent.

Discrete distributions   Consider an example using discrete distributions. Suppose players A, B, and C use strategies given by independent random variables X, Y, and Z, respectively. Suppose X places probability 1 on \$2, Y places probability 1/2 on both \$1 and \$3, while Z places probability 3/4 and 1/4 on \$1 and \$5, respectively. Notice that the mean of each distribution is \$2. Using the notation developed by Merryfield, *et al.* [2], let $W_{A/B}$ be the random variable returning the amount of money that player A wins (or loses) when playing against player B, that is,

$$
W_{A/B} = \begin{cases} -X & \text{if } X > Y \\ Y & \text{if } X < Y \\ 0 & \text{if } X = Y. \end{cases}
$$

If players A and B use strategies X and Y, respectively, X is preferred to Y, denoted $X \succ Y$, if and only if $E(W_{A/B}) > 0$.

Suppose players A and B play the Wallet Game. Player A loses \$2 when player B carries \$1 and wins \$3 when B carries \$3. Player A's expected payoff against player B is $E(W_{A/B}) = \frac{1}{2}(-2) + \frac{1}{2}(3) = \frac{1}{2}$. Thus, strategy X is preferred to strategy Y.

The following matrix, which is similar to one used by Kraitchik [1], is used to compute $E(W_{B/C})$. The $(i, j)^{\text{th}}$ entry of the matrix, m_{ij}, is the amount that player B wins or loses when carrying y_i in his wallet, while player C is carrying z_j in her wallet; this occurs with probability $p_i q_j$.

B/C	$q_0 = 3/4$ $z_0 = 1$	$q_1 = 1/4$ $z_1 = 5$
$p_0 = 1/2 \quad y_0 = 1$	0	5
$p_1 = 1/2 \quad y_1 = 3$	-3	5

Calculating $E(W_{B/C})$ requires summing the products of the matrix entries and their probabilities; in this case,

$$
E(W_{B/C}) = \sum_{i=0}^{1} \sum_{j=0}^{1} p_i q_j m_{ij} = \frac{3}{8}(0) + \frac{3}{8}(-3) + \frac{1}{8}(5) + \frac{1}{8}(5) = \frac{1}{8},
$$

and $Y \succ Z$. Finally, if players A and C play, then $E(W_{A/C}) = \frac{3}{4}(-2) + \frac{1}{4}(5) = -\frac{1}{4}$ and $Z \succ X$.

The lack of transitivity of $X \succ Y$, $Y \succ Z$, and $Z \succ X$ suggests that there may not be a "best" strategy in the discrete case when both players are required to have the same positive mean. The following proposition confirms this, answering the question posed by Merryfield, *et al.* [2], by showing that there is no distribution that is preferred to all others, that is, there is no optimal distribution. (Recall that Y has *finite support* if positive probability is placed on a finite number of values.)

PROPOSITION 1. *For any discrete random variable Y with finite support, there exists a discrete random variable X with $\mu_X = \mu_Y$ such that $X \succ Y$.*

Proof. Suppose player A knows that player B carries an amount of money given by the random variable Y whose probabilities, q_i, are distributed on a finite set of monetary values y_i, such that $y_0 = 0$ and $y_i < y_{i+1}$ for all $i \leq n$. Since the mean of Y, μ_Y, is positive, it follows that $q_0 \neq 1$.

We construct for player A a random variable X that defeats Y. Player A's strategy is to win almost every game; however, when player A loses, she forfeits a large amount of money. Interestingly, player A need only place positive probability on three values, regardless of the complexity of player B's distribution. Define X by the distribution of probabilities p_i on monetary values x_i as follows: $p_0 = q_0$ on $x_0 = 0$, p_1 on $x_1 = \frac{1}{2}y_1$, and $p_2 = 1 - p_0 - p_1$ on x_2, where p_1 and x_2 satisfy the following conditions,

$$\frac{(1 - p_0)\mu_Y}{\mu_Y + \frac{1}{2}(1 - p_0)y_1} < p_1 < 1 - p_0 \quad \text{and} \quad x_2 = \frac{\mu_Y - \frac{1}{2}p_1 y_1}{1 - p_0 - p_1}.$$

Notice that p_1 exists since $p_0 = q_0 \neq 1$. Also, x_2 is defined such that $\mu_X = \mu_Y$.

As in the example above, it is convenient to view the Wallet Game in matrix form. Although we do not know how x_2 compares to the y_is, we assume the worst-case scenario for player A, that is, x_2 is greater than the largest monetary value that player B carries, y_n. As before, the matrix entries are payoffs to player A.

	A/B	q_0 y_0	q_1 y_1	q_2 y_2	q_3 y_3	$\cdots$ $\cdots$	q_n y_n
p_0	x_0	0	y_1	y_2	y_3	$\cdots$	y_n
p_1	$x_1 = y_1/2$	$-x_1$	y_1	y_2	y_3	$\cdots$	y_n
p_2	x_2	$-x_2$	$-x_2$	$-x_2$	$-x_2$	$\cdots$	$-x_2$

The expected values of the first column and first row cancel because

$$p_0(q_1 y_1 + q_2 y_2 + q_3 y_3 + \cdots + q_n y_n) + q_0(-p_1 x_1 - p_2 x_2) = p_0 \mu_Y - q_0 \mu_X = 0.$$

Since $x_1 < y_i$ when $i > 0$, the remaining entries in the second row yield a positive contribution to the expected value for player A in the amount of $p_1(q_1 y_1 + q_2 y_2 + \cdots + q_n y_n)$, or $p_1 \mu_Y$. In this worst-case scenario, $x_2 > y_n$ implies that the remaining entries in the third row contribute the following to the expected value of player A

$$-p_2(q_1 + q_2 + \cdots + q_n)x_2 = -(1 - p_0 - p_1)(1 - q_0)x_2 = -\left(\mu_Y - \frac{1}{2}p_1 y_1\right)(1 - q_0).$$

Hence, we have

$$E(W_{A/B}) \geq p_1 \mu_Y - [\mu_Y - \frac{1}{2} p_1 y_1](1 - q_0) > 0,$$

by the definition of p_1. Therefore, $X \succ Y$. ∎

In our earlier example, and as indicated in the above proposition, playing the mean with probability one can be defeated. However, it *is* a winning strategy against all other symmetric, discrete distributions. In this case, player A loses half of the time with loss μ_X, but wins half of the time with a gain that is greater than μ_X. Hence, the expected payoff is positive. In the next section we focus on continuous density functions and examine the roles of both the mean and median.

Continuous Density Functions   Suppose that random variables X and Y have continuous density functions f and g, respectively. Recall that a continuous density function never places positive probability on a single value; that is, the probability of a player carrying a specific amount of money is zero. As in the discrete case, if g is a symmetric density function then playing the mean with probability one (or equivalently, the median) is preferred to g. Although playing the mean with probability one does not satisfy our restriction to continuous density functions, this idea is easily modified to show the existence of a continuous density function with the same mean (and median) that defeats the original symmetric density function.

We do this in the following proposition, considering nonsymmetric, continuous density functions where players are required to have the same median. Denote the median of the random variable X as m_X. Thus it is equally likely that the player has more than or less than m_X.

PROPOSITION 2. *For any random variable Y with a continuous density function, there exists a random variable X with a continuous density function where $m_X = m_Y$ and $X \succ Y$.*

Proof. Suppose player A knows that player B carries an amount of money given by the random variable Y with probability density function g. The discrete response $X = m_Y$, where m_Y is the median of Y, is preferred to Y. This follows since the median m_Y loses half of the time with a loss of m_Y and wins half of the time, averaging a payoff greater than m_Y. Therefore, the expected payoff for player A is positive. However, this is a discrete distribution. To construct a continuous distribution, playing the median can be considered as the limit of a sequence of uniform distributions where the variances tend to zero. Since the expected value of playing the median is positive, there exists a uniform distribution with $m_X = m_Y$ and positive expected value. ∎

Since there is no optimal continuous density function when the distributions are required to have the same median, let's consider the case where they have the same mean. The following proposition shows that there is no optimal continuous density function in this case either. The proof is constructive, as in the discrete case, and the motivation for the strategy is similar. Once again, Player A's strategy is to win more frequently than player B, while infrequently losing a large sum of money. We construct a density function that matches the opponent on $[0, m_Y]$, and is piecewise uniform on both $[m_Y, m_Y + \epsilon]$ and $[n - \epsilon, n]$, where n and ϵ are selected such that $\mu_X = \mu_Y$ and $X \succ Y$.

PROPOSITION 3. *For any random variable Y with a continuous density function, there exists a random variable X with a continuous density function where $\mu_X = \mu_Y$ and $X \succ Y$.*

Proof. Suppose player A knows that player B carries an amount of money given by the random variable Y with continuous density function g. Suppose g has mean μ_Y and median m_Y. As in the discrete proof, the goal is to construct a density function that defeats g while having the same mean. Let γ be the average conditional expected value of g conditioned on being in the interval $[m_Y, \infty)$, that is, $\gamma = \int_{m_Y}^{\infty} y g(y)\, dy / \int_{m_Y}^{\infty} g(y)\, dy = 2 \int_{m_Y}^{\infty} y g(y)\, dy$.

Let X be a random variable with density function f defined by

$$
f(x) = \begin{cases}
g(x) & \text{on } [0, m_Y] \\
\frac{1-\epsilon}{2\epsilon} & \text{on } (m_Y, m_Y + \epsilon] \\
\frac{1}{2} & \text{on } [n - \epsilon, n] \\
0 & \text{elsewhere},
\end{cases}
$$

where $0 < \epsilon < 1$ is selected so that $n - \epsilon > m_Y + \epsilon$, where

$$
n = \frac{\gamma}{\epsilon} + \epsilon - \frac{1}{2} - \frac{m_Y}{\epsilon} + m_Y,
$$

and so that the following inequality holds:

$$
2\gamma(1 - \epsilon) \int_{m_Y + \epsilon}^{\infty} g(y)\, dy > \left(\gamma + \epsilon^2 - \frac{\epsilon}{2} - m_Y + m_Y \epsilon \right)
$$

$$
+ \ 2(m_Y + \epsilon)(1 - \epsilon) \int_{m_Y}^{m_Y + \epsilon} g(y)\, dy. \qquad (1)
$$

Notice that the left side of (1) converges to γ as ϵ approaches zero, while the right side converges to $\gamma - m_Y$. Also, n grows without bound as ϵ approaches zero. Therefore, a sufficiently small ϵ can be chosen to satisfy both inequalities. Although (1) and the definition of n appear complex, selecting such an ϵ guarantees that $\mu_X = \mu_Y$ and $X \succ Y$ as shown below.

Since f is equal to g on $[0, m_Y]$ and f is composed of piecewise horizontal line segments on (m_Y, ∞), then, by the definition of n,

$$
\mu_X = \int_0^{m_Y} y g(y)\, dy + \left(\frac{1 - \epsilon}{2\epsilon} \right) \epsilon \left(m_Y + \frac{\epsilon}{2} \right) + \frac{1}{2} \epsilon \left(n - \frac{\epsilon}{2} \right)
$$

$$
= \int_0^{m_Y} y g(y)\, dy + \frac{\gamma}{2} = \int_0^{\infty} y g(y)\, dy = \mu_Y.
$$

To see that $X \succ Y$, we employ a matrix again. As in the previous proof, we consider the worst-case scenario for player A. For example, when X is in the interval $[n - \epsilon, n]$ and Y is in $(m_Y + \epsilon, \infty)$, we assume that X loses n. Also, when X is in $[m_Y, m_Y + \epsilon]$ and Y is in $(m_Y + \epsilon, \infty)$ then Y loses, on average, more than γ. In the following matrix, the entries are payoffs to player A. Let γ_1 and γ_2 be the average conditional expected values of g conditioned on being in $(m_Y, m_Y + \epsilon]$ and $(m_Y + \epsilon, \infty)$, respectively.

A/B	$\frac{1}{2}$ $[0, m_Y]$	$\int_{m_Y}^{m_Y + \epsilon} g(y)\, dy$ $(m_Y, m_Y + \epsilon]$	$\int_{m_Y + \epsilon}^{\infty} g(y)\, dy$ $(m_Y + \epsilon, \infty)$
$\frac{1}{2}$ $\quad [0, m_Y]$	0	γ_1	γ_2
$\frac{1-\epsilon}{2}$ $\quad [m_Y, m_Y + \epsilon]$	$-(m_y + \frac{\epsilon}{2})$	$-(m_Y + \epsilon)$	γ
$\frac{\epsilon}{2}$ $\quad [n - \epsilon, n]$	$-(n - \frac{\epsilon}{2})$	$-n$	$-n$

The contribution to the expected value from the first row is

$$\frac{\gamma_1}{2}\left(\int_{m_Y}^{m_Y+\epsilon} g(y)\,dy\right) + \frac{\gamma_2}{2}\left(\int_{m_Y+\epsilon}^{\infty} g(y)\,dy\right) = \frac{\gamma}{4},$$

which cancels with the contribution from the first column since

$$\frac{\gamma}{4} - \left(\frac{1-\epsilon}{4}\right)\left(m_Y + \frac{\epsilon}{2}\right) - \frac{\epsilon}{4}\left(n - \frac{\epsilon}{2}\right) = 0,$$

by definition of n. Computing the contribution to the expected payoff from the remaining entries of the matrix, player A wins (or loses)

$$\frac{-n\epsilon}{4} - (m_Y + \epsilon)\left(\frac{1-\epsilon}{2}\right)\int_{m_Y}^{m_Y+\epsilon} g(y)\,dy + \gamma\left(\frac{1-\epsilon}{2}\right)\int_{m_Y+\epsilon}^{\infty} g(y)\,dy. \quad (2)$$

Substituting for n, (2) is positive if

$$2\gamma(1-\epsilon)\int_{m_Y+\epsilon}^{\infty} g(y)\,dy > \left(\gamma + \epsilon^2 - \frac{\epsilon}{2} - m_Y + m_Y\epsilon\right)$$

$$+ 2(m_Y + \epsilon)(1-\epsilon)\int_{m_Y}^{m_Y+\epsilon} g(y)\,dy.$$

This inequality holds by the selection of ϵ. Therefore, $X \succ Y$.   ∎

Game-theoretic conclusion   Let's interpret the propositions in this paper game-theoretically. A pair of strategies is a Nash equilibrium if neither player, given knowledge of her opponent's strategy, can improve her outcome by deviating from her strategy. Since the Wallet Game is a zero-sum game, at least one player must have a nonpositive expected payoff. Using the constructions in the propositions, this player can change her (discrete or continuous) distribution to yield a positive expected payoff. So, there does not exist a Nash equilibrium in any of the cases we considered.

   In game theory, the fundamental solution concept is the Nash equilibrium. Consequently, the fact that there is no optimal strategy, hence no Nash equilibrium, may seem troubling. It is interesting to note that while the existence of Nash equilibria is often proved by variations or extensions of the Kakutani Fixed Point Theorem, this theorem does not apply here as the hypotheses require the set of strategies to be compact [4]. Neither the space of all discrete random variables with fixed means nor the space of all continuous distributions with fixed medians or means are compact.

   So what should you do when someone suggests playing the Wallet Game? Since the standard game-theoretic assumption of knowing your opponent's strategy is highly unlikely, the authors advise readers to play the game at their own risk.

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